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PSEUDO FORCE–STATE MAPPING METHOD: INCORPORATION OF THE EMBEDDING AND FORCE–STATE MAPPING METHODS

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1. INTRODUCTION

The force–state mapping method is one of the widely used system identification methods for non-linear mechanical systems. It is a surface representation of net force versus restoring forces [1,2]. As an example, the equation of motion of a SDOF mechanical system can be expressed as

$$f(x, \dot{x}) = F(t) - m\ddot{x} = F_{net}, \tag{1}$$

where $f(x, \dot{x})$ is a function which denotes the restoring force of the system and F_{net} is the net force. If the applied force, mass, acceleration signal and the state variables (x and $\dot{x}$) are measured, one can construct a three-dimensional plot of F_{net} versus x and $\dot{x}$. The state variables can be found by direct measurements or through integration of measured acceleration $\ddot{x}$. The direct measurement of all the state variables is usually a problem, and creation of displacement x from two successive numerical integrations of $\ddot{x}$ may cause significant drifting effects. One can use numerical differentiation once x is measured, but this method suffers from the effects of noise on the measurement. In this letter, an alternative method is suggested which may be used when all the state variables are not available and when exact values of the system parameters do not need to be estimated. The state space characterized by x and $\dot{x}$ is the phase space, and using Takens' Embedding theorem [3], one can reconstruct the phase portrait from a single measured state variable by using delay co-ordinates (method of delays); i.e., a point on the reconstructed phase portrait (attractor) is given by an n-dimensional vector $\mathbf{y}(t) = \{v(t), v(t+T), \ldots, v(t+(n-1)T)\}$, where v are the measured scalar time series and T is the arbitrary chosen delay time. The reconstructed phase portrait is often called the pseudo phase portrait, and this embedding procedure generally requires that $n \geqslant 2k + 1$, where k is the dimension of the original attractor of the dynamical system where the original attractor is an m-dimensional smooth compact manifold. (Note that the dimension of the system is the number of state variables, and the dimension of the attractor is the geometrical dimension of the attractor itself in the phase space. Thus, the dimension of the attractor can be varied according to the solution of the system.) One can find the minimum embedding dimension as described in [4–6]. As an example, the dimension of a circle is one, and the circle can reside in a two-dimensional space which is less than that required by the criterion. For the mechanical systems considered in this letter, the two-dimensional original phase space may be mapped to a two-dimensional pseudo phase space as can be seen in the next section. In this case, one can reconstruct the three-dimensional pseudo force–state map by using the embedding method (method of delays), if the net force and one of the state variables in equation (1) are known.

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2. PSEUDO FORCE–STATE MAPPING METHOD

The term "pseudo" means that the method is based on the embedding process, and it is analogous to the "pseudo phase space" which is reconstructed from a time series. For reconstruction of the phase portrait, one can use the method of delays mentioned earlier or can also use SVD (Singular Value Decomposition) [7]. Both methods can be used for the pseudo force–state map, and extensive study and results using these methods can be found in [8]. In this letter, only the method of delays is considered. For a description of the embedding process, a displacement signal (x) is used for convenience, although any one of the state variables can be used. Let X be a set of all points of a function $f_1(x(t), \dot{x}(t))$ for a SDOF mechanical system (1), and Y be a set of all points of a function $f_2(x(t), x(t-T))$. If $\dot{x}(t)$ is approximated by a simple backward difference, $\dot{x}(\mathrm{t}) = [x(t) - x(t-T)]/T$, there is a function g such that g: $X \to Y$ is one-to-one and invertible. Thus the relationship between f_1 and f_2 can be written as

$$g(f_1(x(t), \dot{x}(t)) = f_2(x(t), x(t-T)). \tag{2}$$

Thus, equation (1) may be modified as

$$f(x(t), x(t-T), T) = F(t) - m\ddot{x} = F_{net}(t). \tag{3}$$

Note that we include the delay time T explicitly in the function to emphasize the dependence on T. Although the phase portrait reconstructed by the embedding process is topologically equivalent to the original, the quality of function estimated from the surface generated by equation (3) must depend on T; i.e., the quality of the surface may be dependent on the delay time used.

As an example, a SDOF linear system is first considered:

$$f_1(x, \dot{x}) = kx + c\dot{x} = F(t) - m\ddot{x} = F_{net}. \tag{4}$$

The original phase space, $f_1(x, \dot{x})$, may be mapped to the delayed co-ordinates as

$$f_1(x, \dot{x}) = kx + c\dot{x} \to \left(k + \frac{c}{T}\right)x(t) - \frac{c}{T}x(t-T) = k'x(t) + c'x(t-T)$$

$$= f_2(x(t), x(t-T)) \tag{5}$$

if a backward difference is assumed. In this case, the function g that maps f_1 to f_2 becomes

$$g(f_1(x, \dot{x})) = f_2(x(t), x(t-T)): \quad (k, c) \to \left(k + \frac{c}{T}, -\frac{c}{T}\right). \tag{6}$$

The original force–state map for $c = 0{\cdot}5$, $k = 1$ and $m = 1$ is shown in Figure 1(a), and Figure 1(b) shows the pseudo force–state map for this system. The delay time is $T = 16T_s$', where $T_s = 0{\cdot}1$ s and is the sampling time. The choice of optimum delay time for the reconstruction of phase portrait can be found in a number of references [6, 9–11]. From these figures, it can be shown that the planar surface of the original force–state map remains plane in the pseudo force–state map.

For the next example, a cubic stiffness term is included to introduce a non-linearity; i.e.,

$$f_1(x, \dot{x}) = kx + \varepsilon x^3 + c\dot{x} = F(t) - m\ddot{x} = F_{net}. \tag{7}$$

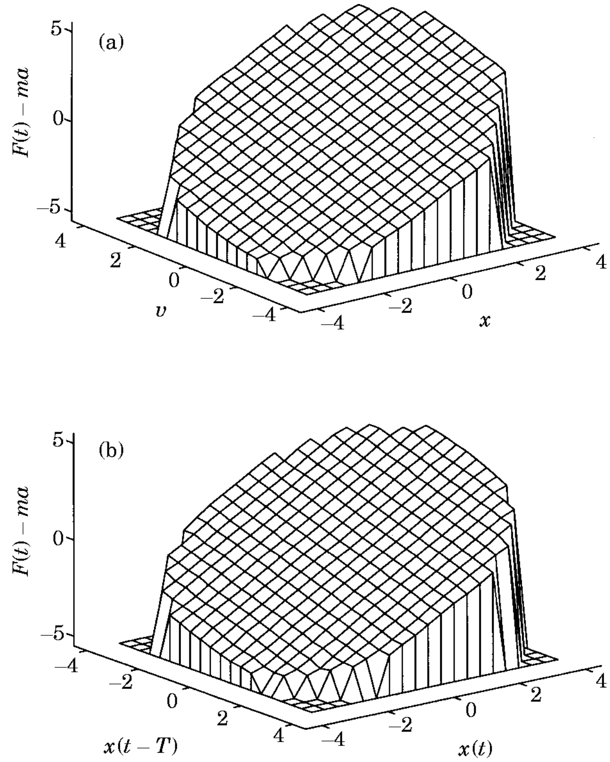

Figure 1. (a) Force–state map of the linear system (4) for $c = 0{\cdot}5$, $k = 1$ and $m = 1$: $F(t) - ma$ versus x and v, where $v = \dot{x}$ and $a = \ddot{x}$. (b) Pseudo force–state map of the linear system (4) for $c = 0{\cdot}5$, $k = 1$, $m = 1$, $T = 16T_s$ and $T_s = 0{\cdot}1$ s: $F(t) - ma$ versus $x(t)$ and $x(t - T)$.

Similar to the previous example, the original phase space, $f_1(x, \dot{x})$, may be mapped to the delayed co-ordinates as

$$f_1(x, \dot{x}) = kx + \varepsilon x^3 + c\dot{x} \rightarrow \left(k + \frac{c}{T}\right)x(t) + \varepsilon x^3(t) - \frac{c}{T}x(t - T)$$

$$= k'x(t) + \varepsilon' x^3(t) + c'x(t - T) = f_2(x(t), x(t - T)) \quad (8)$$

and

$$g(f_1(x, \dot{x})) = f_2(x(t), x(t - T)): \quad (k, \varepsilon, c) \rightarrow \left(k + \frac{c}{T}, \varepsilon, -\frac{c}{T}\right). \quad (9)$$

The original force–state map for $c = 0{\cdot}5$, $k = 1$, $\varepsilon = 1$ and $m = 1$ is shown in Figure 2(a), and the pseudo force–state map is shown in Figure 2(b). From these figures, it is shown that the cubic non-linearity in the original force–state map carries over in the pseudo force–state map. If parameters (k', ε', c') are estimated from the pseudo force–state map, the original system parameters (k, ε, c) can be obtained since the function g is invertible; i.e.,

$$\begin{pmatrix} c \\ k \\ \varepsilon \end{pmatrix} = \begin{pmatrix} -T & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c' \\ k' \\ \varepsilon' \end{pmatrix}. \quad (10)$$

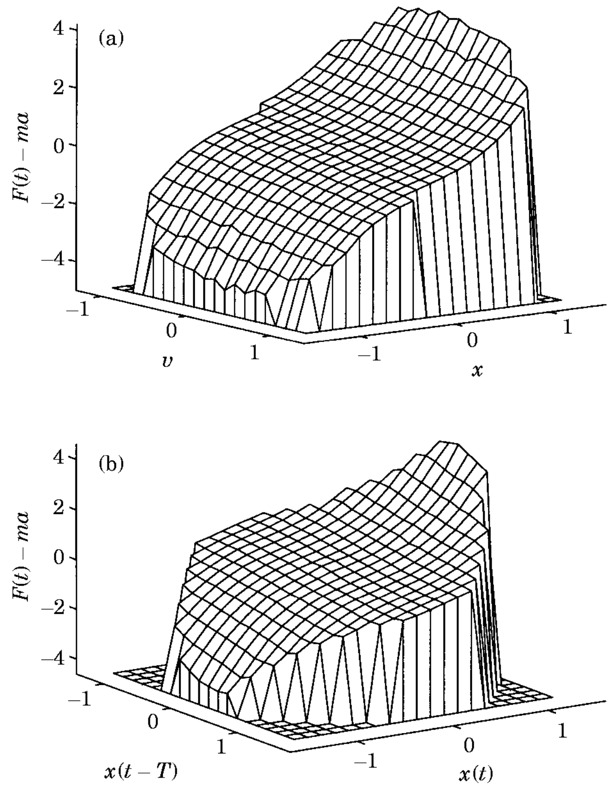

Figure 2. (a) Force–state map of the linear system (7) for $c = 0{\cdot}5$, $k = 1$, $\varepsilon = 1$ and $m = 1$: $F(t) - ma$ versus x and v, where $v = \dot{x}$ and $a = \ddot{x}$. (b) Pseudo force–state map of the non-linear system (7) for $c = 0{\cdot}5$, $k = 1$, $\varepsilon = 1$, $m = 1$, $T = 16T_s$ and $T_s = 0{\cdot}1$ s: $F(t) - ma$ versus $x(t)$ and $x(t - T)$.

However, one must be careful when using this relationship. The parameters obtained from equation (10) have meaningful results only when the delay time T is very small, since it is approximated by a simple backward difference scheme. In practice, the delay time T is chosen to be large enough to cover wide range of pseudo phase space, and so equation (10) may not be successful in this case.

3. CONCLUDING REMARKS

From the above two examples, it is shown that the "embedding method" can be incorporated with the force–state mapping technique, and the resulting surface form of the pseudo force–state map preserves the structure of the original non-linearity of the system. Thus, the pseudo force–state mapping method may be used in a practical situation when one cannot obtain all the state variables; i.e., if one obtains one of the state variables (displacement or velocity) and fails to differentiate or to integrate the obtained signal to obtain the other state variable, this method will be a good alternative. The pseudo force–state mapping method may be extended for multi-degree-of-freedom systems by using the linearized normal modes described in references [12, 13]. Also, the invertibility of g may be generalised for various delay times T, although this is beyond the scope of this letter.

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